Beyond Locality-Sensitive Hashing

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Brown ICERM Theory Seminar

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- Let P be an n-point subset of a metric (X, D), $r > 0$
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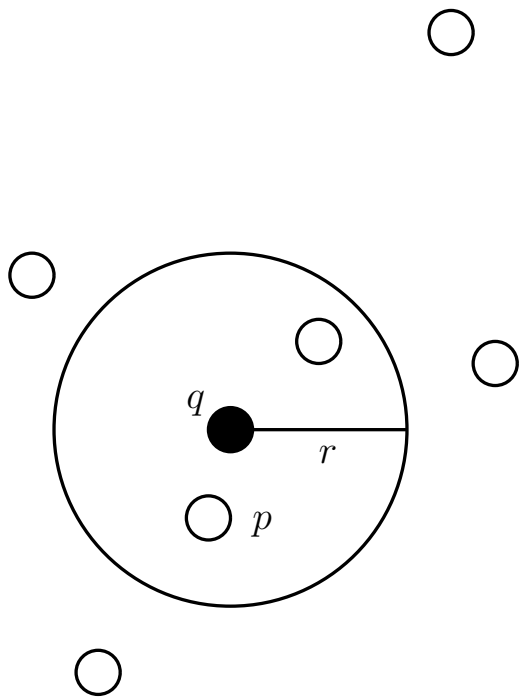

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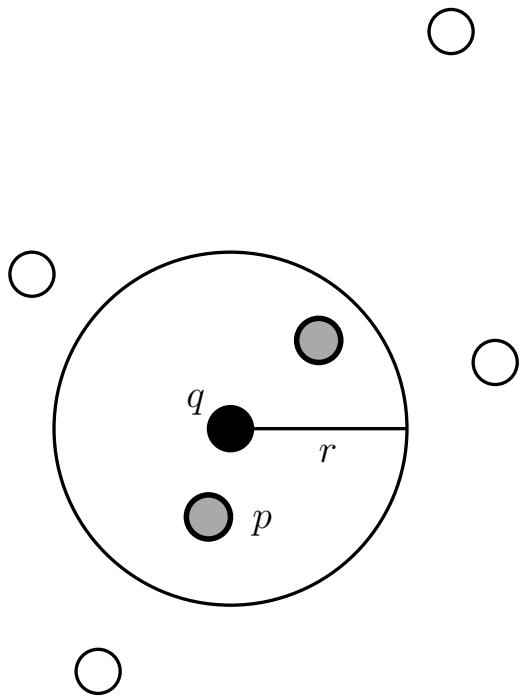

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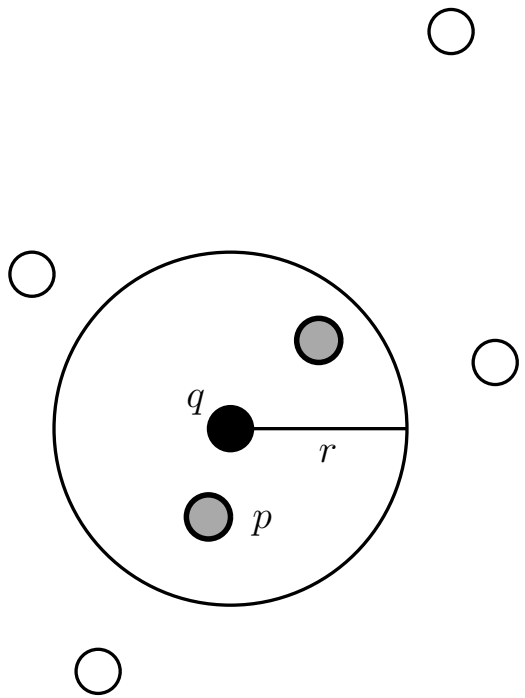
- For $q \in X$ find any $p \in P$ with $D(p, q) \leq r$
- Hard, if (X, D) is *high-dimensional* (space or query time is exponential in the dimension)



Applications

- Machine Learning / Data Mining
- Spell checkers
- Computational Geometry
- Databases
- Clustering
- You name it ...

The Approximate Near Neighbor Problem (ANN)

- Let P be an n-point subset of a metric (X, D), $r > 0$, $c > 1$
- For $q \in X$ find any $p' \in P$ with $D(p', q) \leq cr$, provided that there exists $p \in P$ with $D(p, q) \leq r$

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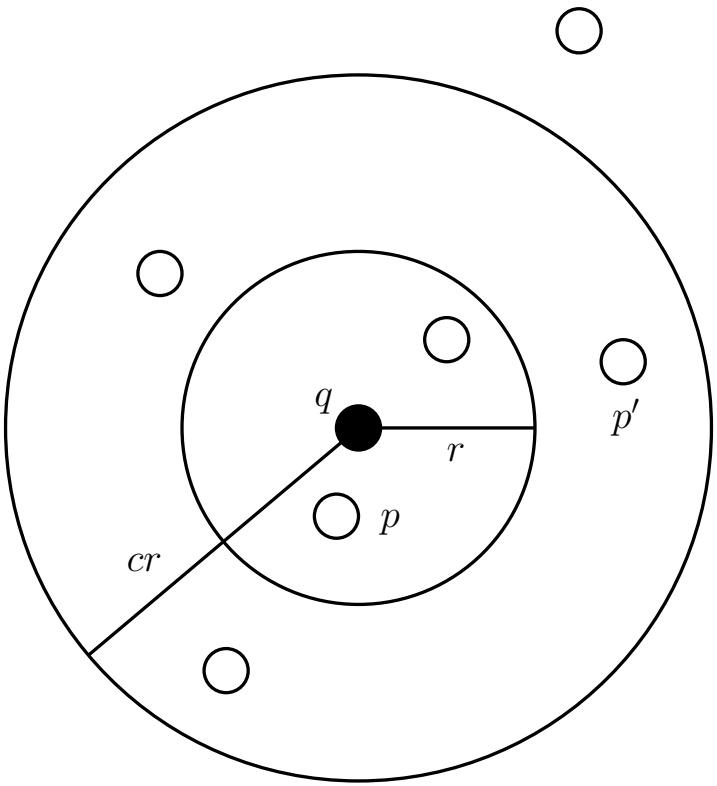

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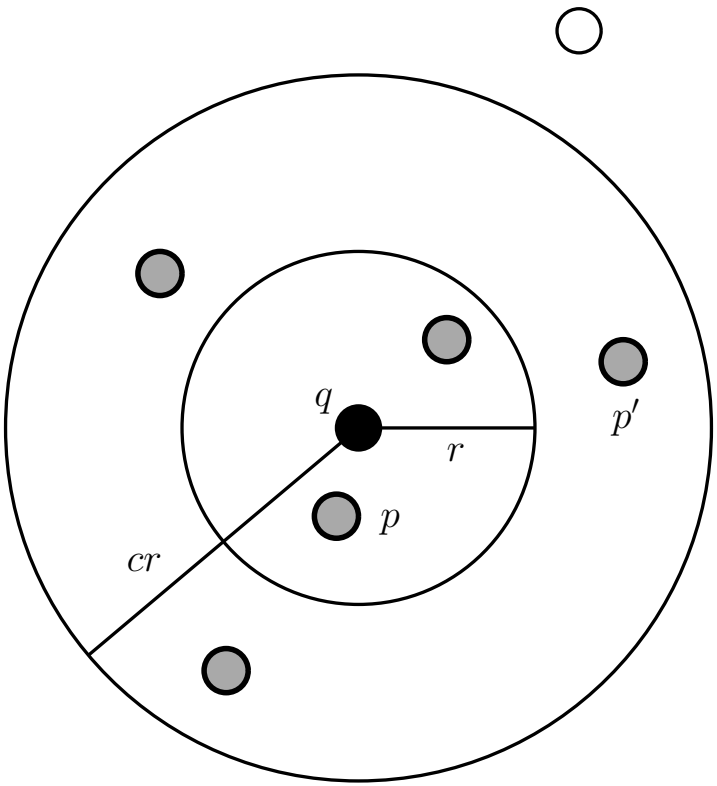

Literature

- **Exponential dependence on the dimension:**
  (Arya, Mount 1993), (Meister 1993), (Clarkson 1994),
  (Arya, Mount, Netanyahu, Silverman, We, 1998), (Kleinberg, 1997),
  (Har-Peled 2002)
- **Polynomial dependence on the dimension:**
  (Indyk, Motwani 1998), (Kushilevitz, Ostrovsky, Rabani 1998),
  (Indyk 1998), (Indyk 2001), (Gionis, Indyk, Motwani 1999),
  (Charikar 2002), (Datar, Immorlica, Indyk, Mirrokni 2004),
  (Chakrabarti, Regev 2004), (Panigrahy 2006), (Ailon, Chazelle 2006),
  (Andoni, Indyk 2006), (Indyk, Kapralov 2013), (Nguyễn 2013),
  (Andoni 2014)

Two regimes

What is known about c-approximate ANN in $\mathbb{R}^d$?

approximation	space	query time
$c = 1 + \varepsilon$	$(dn)^{O_\varepsilon(1)}$	$(d \log n/\varepsilon)^{O(1)}$
$c \to \infty$ $\rho = \rho(c) \to 0$	$d^{O(1)} \cdot n^{1+\rho+o(1)}$	$d^{O(1)} \cdot n^{\rho+o(1)}$

This talk is exclusively about the second regime

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  - if $D(p, q) \leq r$, then $\Pr_{h \sim \mathcal{H}}[h(p) = h(q)] \geq p_1$;
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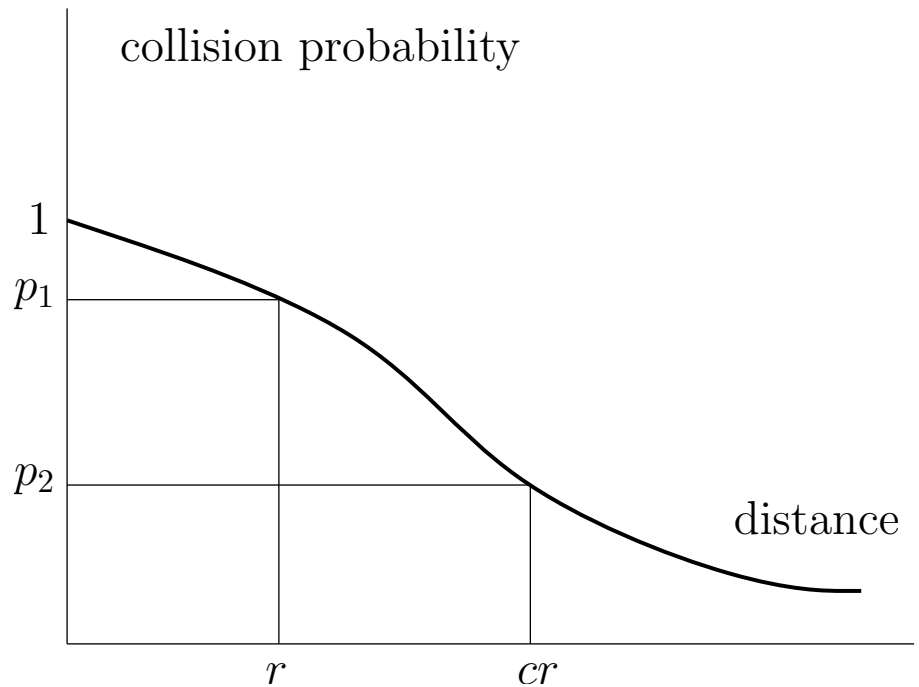

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Known LSH constructions

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Bounds on $\rho = \ln(1/p_1)/\ln(1/p_2)$ for various spaces:

Space	Upper bound	Lower bound
ℓ_1	$\rho \leq 1/c$	$\rho \geq 1/c - o(1)$
ℓ_p $1 < p < 2$	$\rho \leq 1/c^p + o(1)$	$\rho \geq 1/c^p - o(1)$
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This work: ANN in space $O(n^{1+\tau} + nd)$ and time $O(dn^\tau)$, where

- $\tau \leq \frac{7}{8c} + O\left(\frac{1}{c^{3/2}}\right) + o(1)$ for ℓ_1
- $\tau \leq \frac{7}{8c^2} + O\left(\frac{1}{c^3}\right) + o(1)$ for ℓ_2

The first improvement upon (Indyk, Motwani 1998) for ℓ_1 and (Andoni, Indyk 2006) for ℓ_2!

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- Parallels with practice!
  - PCA trees (Sproull 1991), (McNames 2001), (Verma, Kpotufe, Dasgupta 2009)
  - Spectral Hashing (Weiss, Torralba, Fergus 2008)
  - Semantic Hashing (Salakhutdinov, Hinton 2009)
  - WTA Hashing (Yagnik, Strelow, Ross, Lin 2011)

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- Partition P into low-diameter clusters (of diameter $O(cr)$)
- Improve upon $1/c^2$ for the low-diameter case

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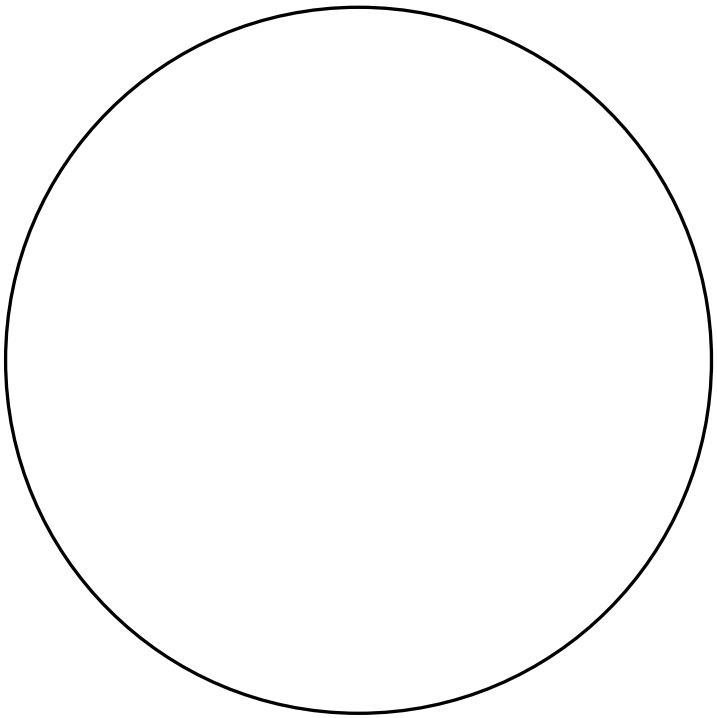

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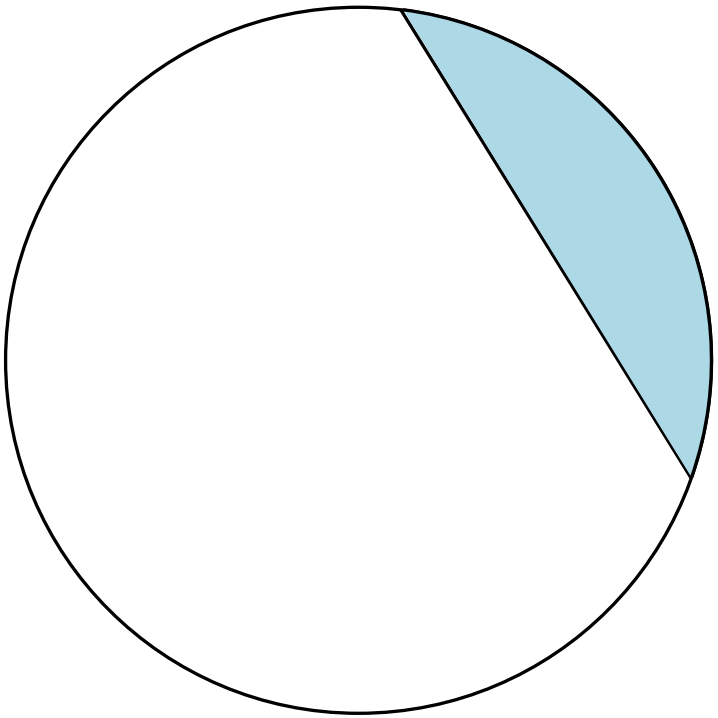

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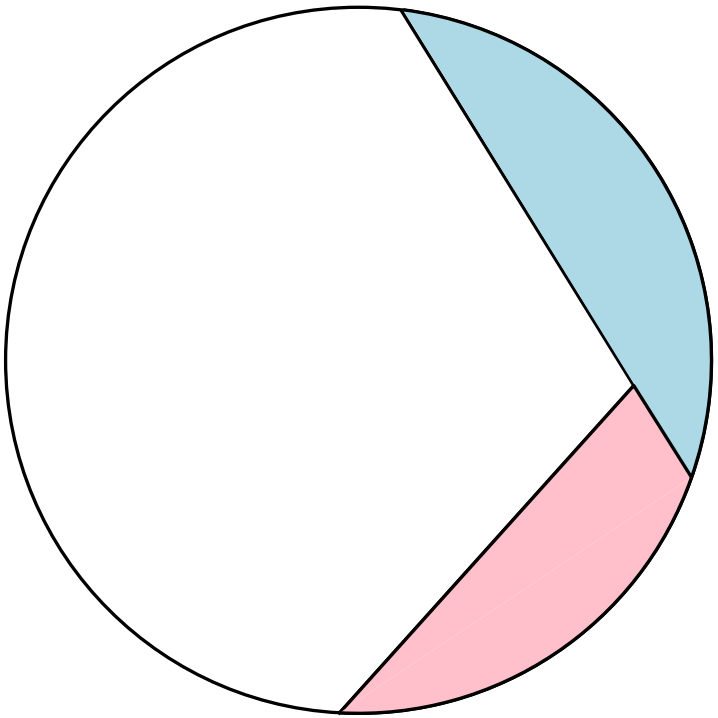

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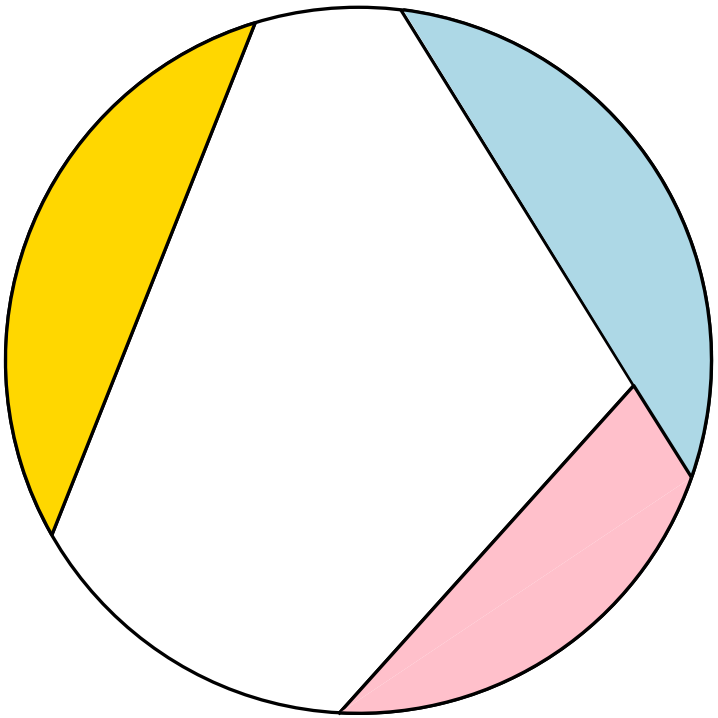

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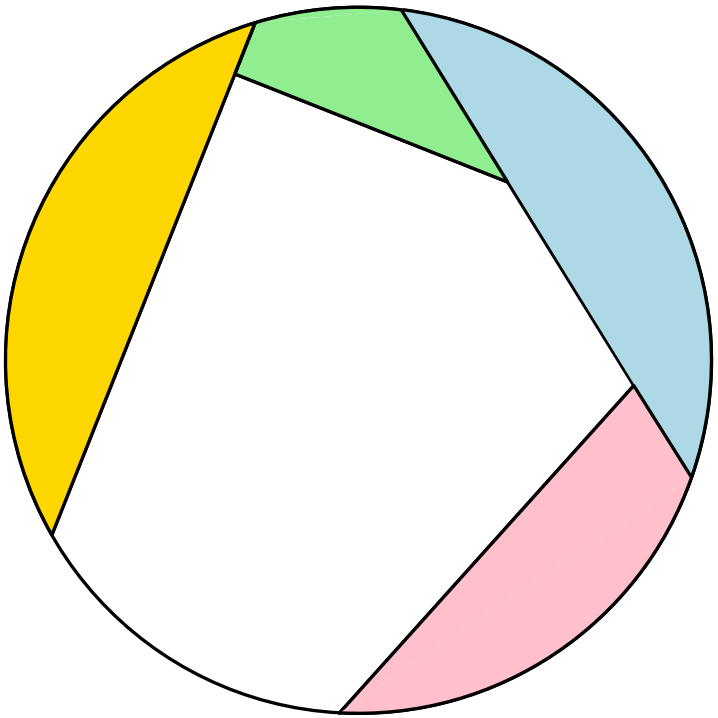

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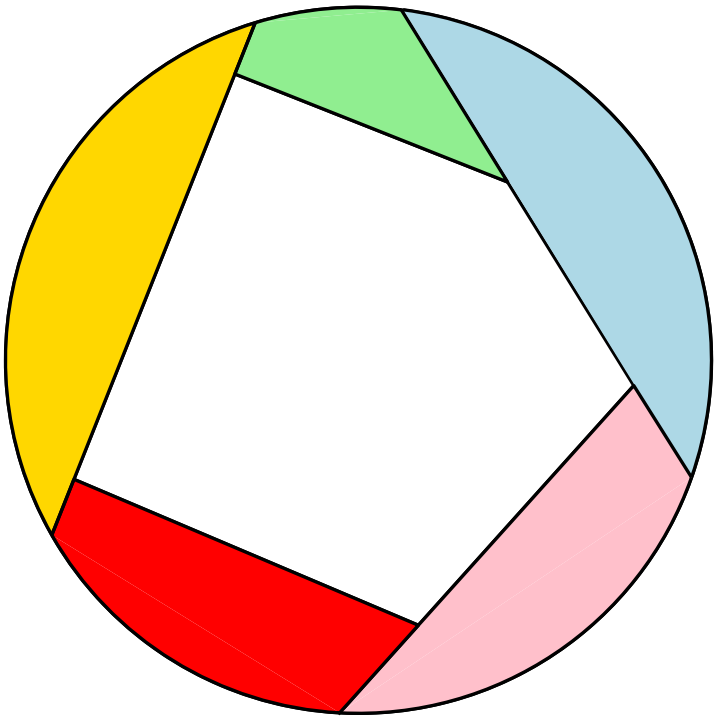

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- Sample a point x wrt to a measure μ, "carve" a ball $B(x, R)$, repeat until everything is covered

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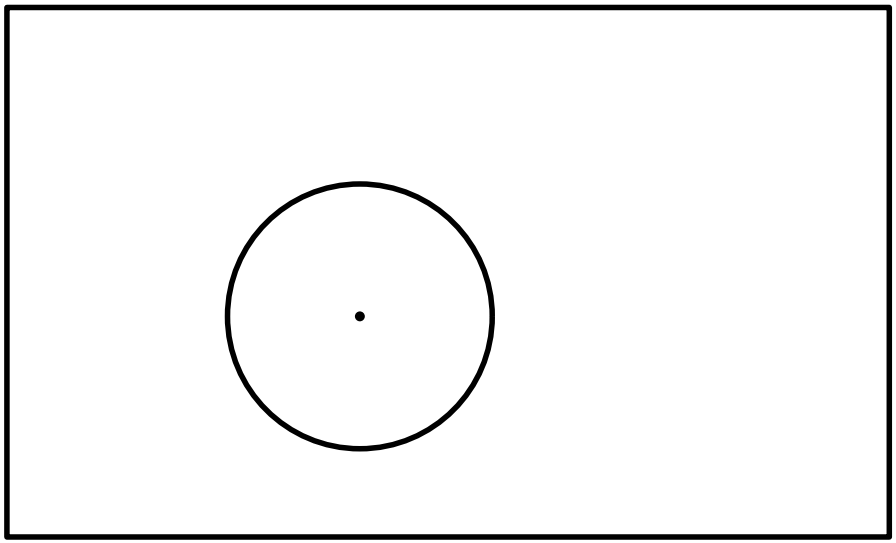

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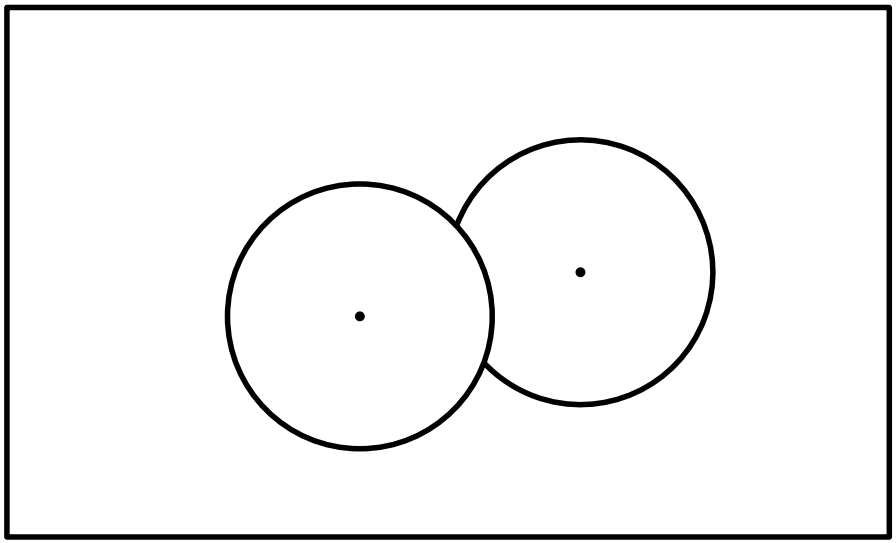

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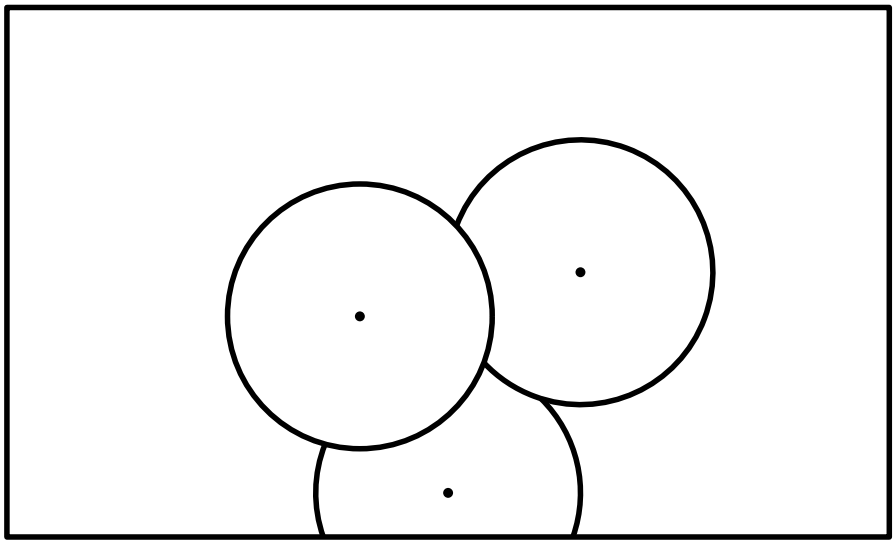

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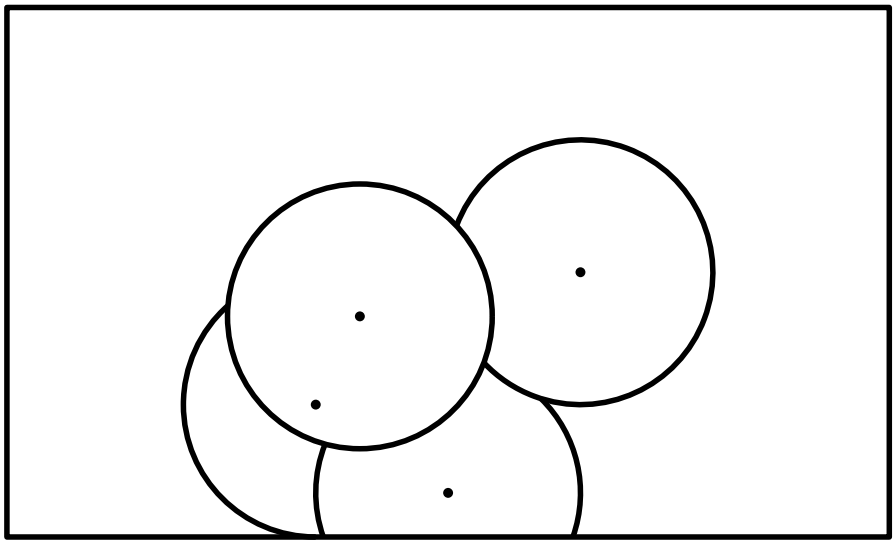

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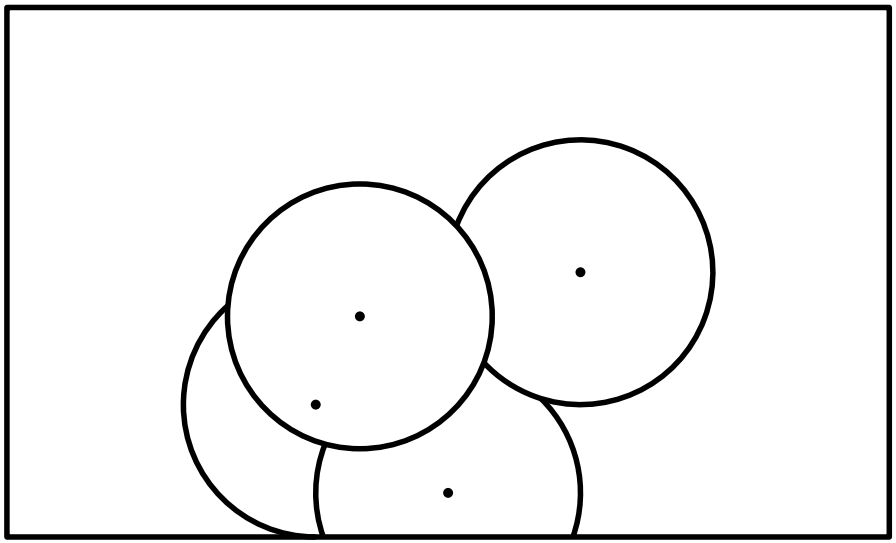

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- Probability of collision?

$$\Pr[x \text{ and } y \text{ collide}] = \frac{\mu(B(x,R) \cap B(y,R))}{\mu(B(x,R) \cup B(y,R))}$$

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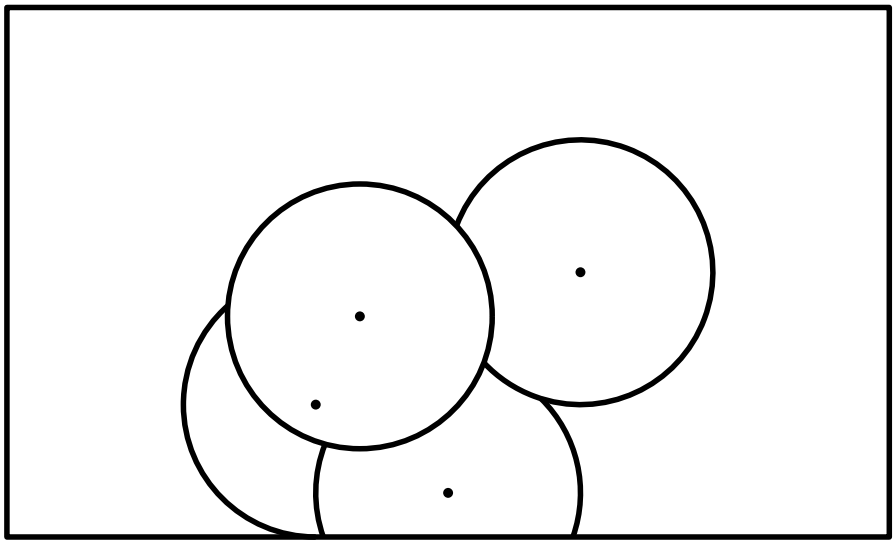

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- Number of balls $\approx \mu(X)/\mu(B(x,R))$

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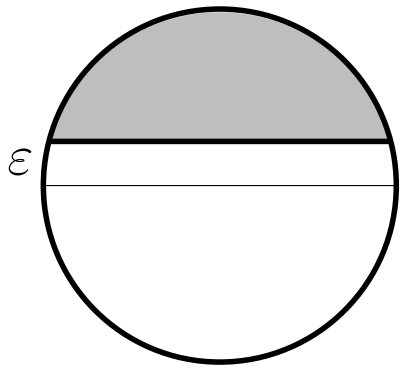

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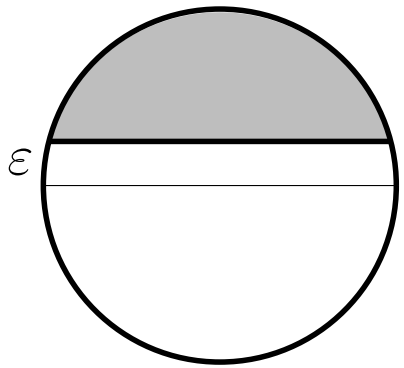

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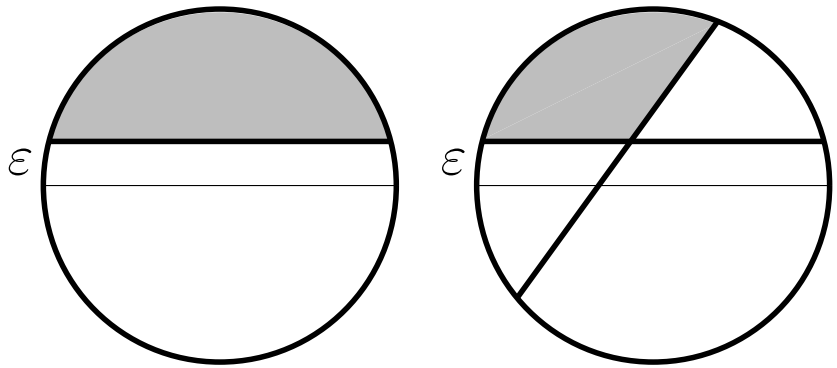

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- Computing the intersection of two caps is non-trivial...

Gaussian LSH

- Sample $w \sim N(0,1)^d$
- Carve $\left\{u \mid \|u\| = \eta cr, \langle u, w\rangle \geq \varepsilon\sqrt{d} \cdot \eta cr\right\}$, repeat
- The same as in (Karger, Motwani, Sudan 1998)

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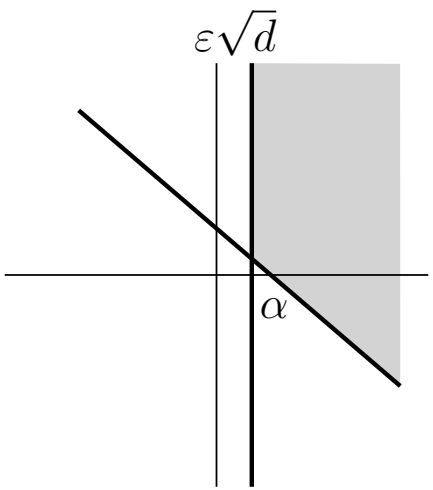
$$\Pr_{X,Y\sim N(0,1)}[X \geq \varepsilon\sqrt{d} \wedge \cos\alpha \cdot X - \sin\alpha \cdot Y \geq \varepsilon\sqrt{d}],$$

where α is the angle between points of interest



Gaussian LSH: the result

- Easy computations show that this family for $\varepsilon = d^{-1/4}$ is (r, cr, p_1, p_2)-sensitive with
  - $p_1, p_2 = \exp(-o(d))$ $(= n^{-o(1)})$
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$$\rho = \frac{\ln(1/p_1)}{\ln(1/p_2)} \approx \left(1 - \frac{1}{4\eta^2}\right) \cdot \frac{1}{c^2} = \frac{1 - \Omega_\eta(1)}{c^2}$$

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- Simplification and improvement of the result from (Andoni, Indyk 2006) for the case, when everything lies in a ball of radius $O(cr)$!

From LSH to ANN: the basic reduction

- Let $\mathcal{H}$ be a (r, cr, p_1, p_2)-sensitive family: for every $p, q \in X$
  - if $D(p, q) \leq r$, then $\Pr_{h \sim \mathcal{H}}[h(p) = h(q)] \geq p_1$;
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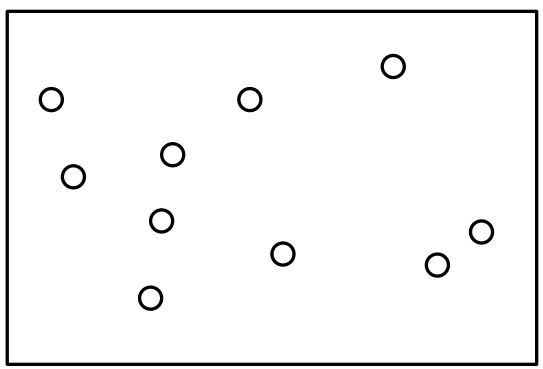

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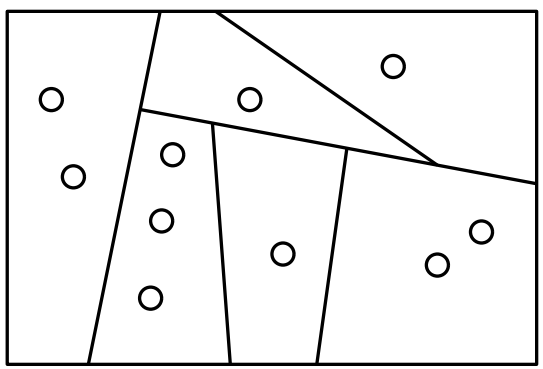

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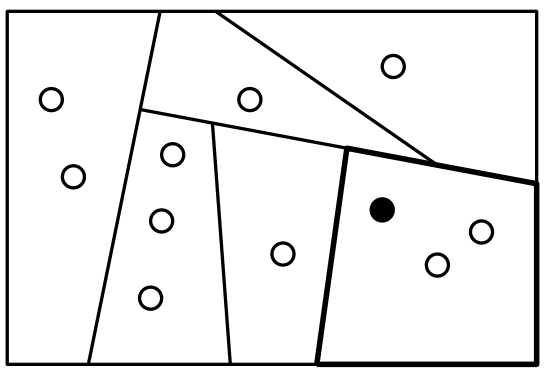

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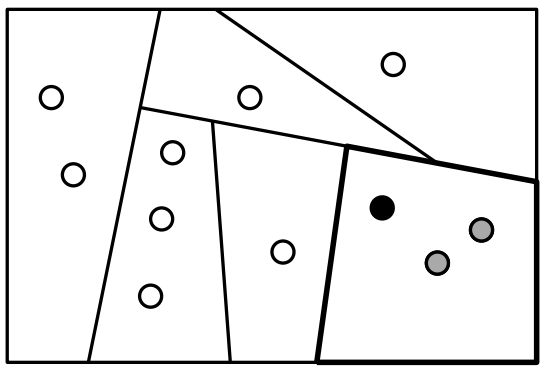
- The optimal choice of k leads to the need in n^ρ hash tables
- Overall: $n^{1+\rho}$ space, n^ρ query time



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- Choose k s.t. for every $p, q \in X$
  - if $D(p,q) \leq r$, then $\Pr_{h\sim\mathcal{H}^{\otimes k}}[h(p) = h(q)] \geq n^{-\rho-o(1)}$;
  - if $D(p,q) \geq cr$, then $\Pr_{h\sim\mathcal{H}^{\otimes k}}[h(p) = h(q)] \leq 1/n$,

  where $\rho = \ln(1/p_1)/\ln(1/p_2)$.

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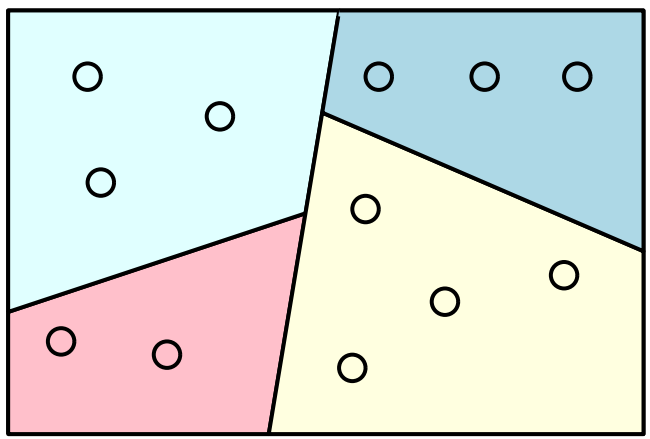
- Consider $n^{\rho+o(1)}$ independent hash tables (each uses its own $h \sim \mathcal{H}^{\otimes k}$)
- Space $\approx n^{1+\rho+o(1)}$, query time $\approx n^{\rho+o(1)}$

From LSH to ANN: two-level hashing

- Partition space somewhat coarsely (using smaller k than before)



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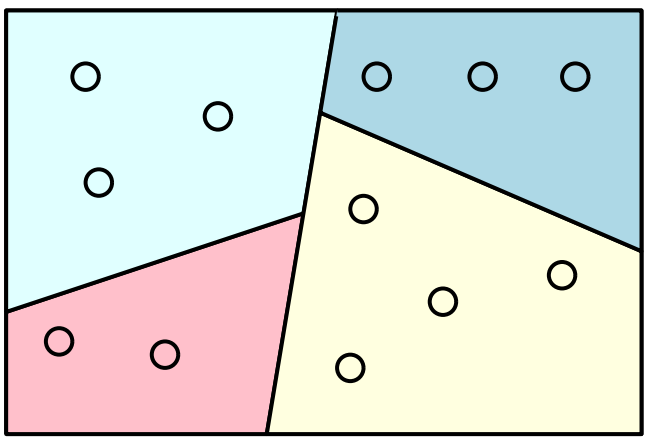

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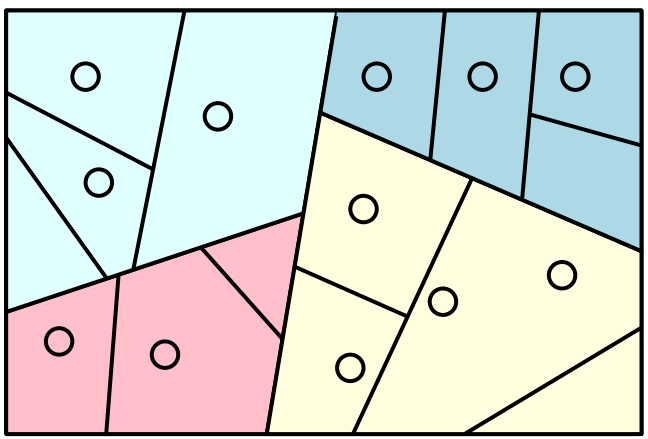

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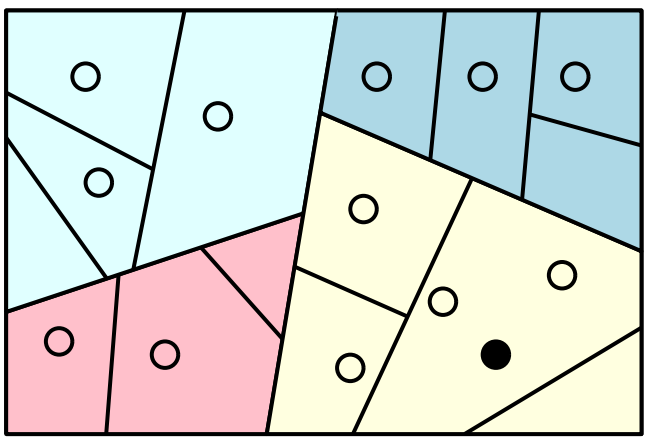

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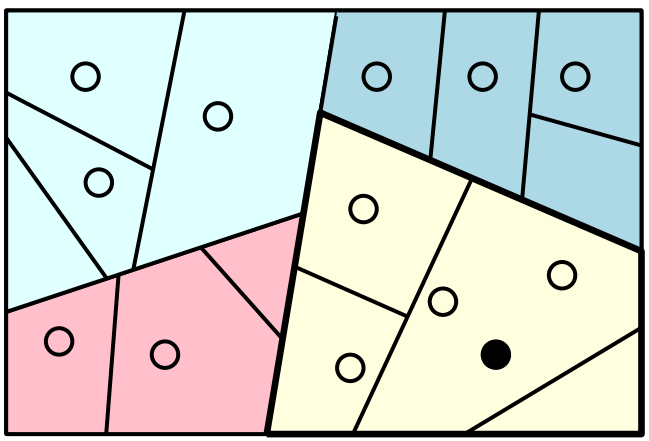

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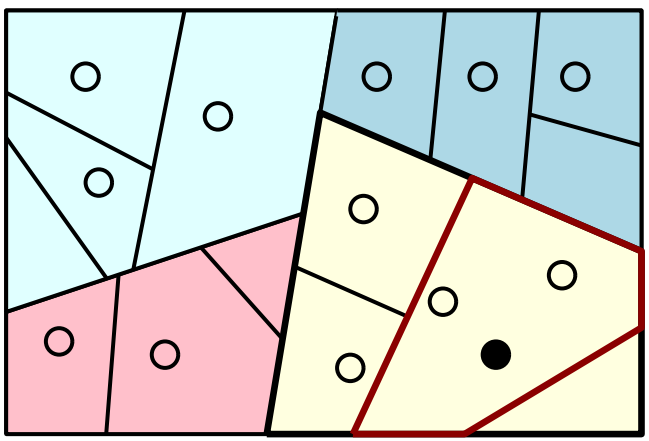

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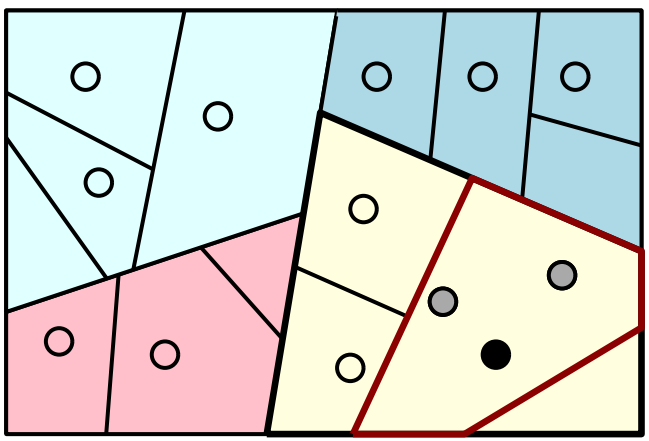

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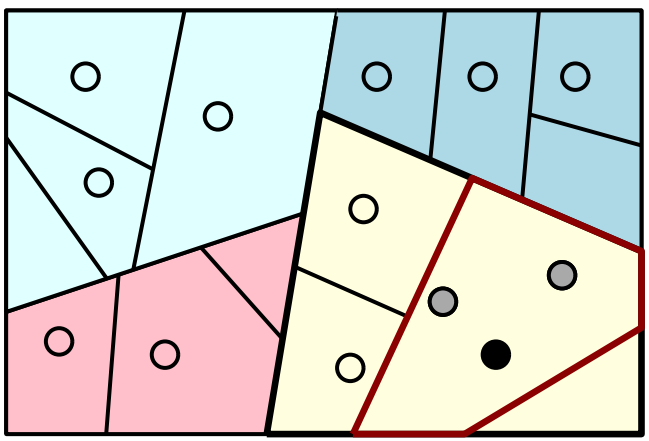
- Get $\rho \leq (1 - \Omega(1))/c^2$, since in inner tables we get better relation between p_1 and p_2!



The proof idea

- Let $\tau > 1$ be some parameter
- Let $\mathcal{H}_1$ be the LSH family for ℓ_2 that achieves $\rho \lesssim 1/c^2$, choose k s.t. the probabilities of collision for $\mathcal{H}_1^{\otimes k}$ are as follows

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- Due to the independence the collision probabilities for one scale multiply (kind of), as a result obtain $\rho \leq (1 - \Omega_\tau(1))/c^2$

Reflections

Why is this family data-dependent?

- Use a point from P as a center for an inner hash table
- If $q \in X$ is far from the center of the outer bin, then we can not handle it (but we do not care about this case)

Smallest enclosing balls

- Jung's theorem: any set of *diameter* D lies in a ball of *radius* $D/\sqrt{2}$
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- Careful analysis leads to

$$\rho \leq \frac{7}{8c^2} + O\left(\frac{1}{c^3}\right) + o_c(1)$$

Extensions

- Can embed ℓ_1 into ℓ_2-squared, which gives an algorithm with

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  for the Euclidean case (and the similar result for ℓ_1 and Hamming)
- Using this multilevel partitioning can improve known constructions for spanners for subsets of ℓ_1 and ℓ_2
  (upon (Har-Peled, Indyk, Sidiropoulos 2013))

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- Can one improve the bound for this random instance further? (Looks hard!)