Cutting modulus

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The goal of this document is to define the *cutting modulus* of a metric space (X, d_X). Let us start with a few auxiliary definitions.

For a finite set U, denote $\Delta(U)$ to be the set of all probability measures over U, and denote $\Gamma(U)$ to be the set of all *symmetric* probability measures over $U \times U$. Abusing notation, for $\nu \in \Delta(U)$ and $u \in U$, we write $\nu(u) = \nu(\{u\})$; similarly, for $\mu \in \Gamma(U)$ and $u, v \in U$, we write $\mu(u,v) = \mu\Big(\{(u,v)\}\Big)$. For $\mu \in \Gamma(U)$, we denote $\rho_\mu \in \Delta(U)$ to be the following *marginal* measure:

$$\rho_\mu(u) = \sum_{v \in U} \mu(u,v).$$

For $\mu \in \Gamma(U)$ and $S \subset U$ with $0 < \rho_\mu(S) < 1$, we define the conductance of S with respect to μ as follows:

$$\Phi_\mu(S) = \frac{\mu\big(S \times (U \setminus S)\big)}{\min\{\rho_\mu(S), 1 - \rho_\mu(S)\}}.$$

If the above denominator is zero, we define $\Phi_\mu(S) = +\infty$.

Let (X, d_X) be a *finite* metric space. For $r \geq 0$, we denote $\Gamma_r(X) \subseteq \Gamma(X)$ to be the set of all symmetric probability measures over $X \times X$ supported on the pairs of points within d_X-distance at most r.

Definition 0.1. *We say that a probability measure $\nu \in \Delta(X)$ is* (R, β)-dispersed *for $R > 0$ and $0 \leq \beta \leq 1$ if for every point $x_0 \in X$, one has $\nu\big(B_X(x_0, R)\big) \leq \beta$.*

We denote the set of all the (R,β)-dispersed probability measures over X by $\Delta_{R,\beta}(X)$.

We may now formally define the notion of cutting modulus of (X, d_X).

Definition 0.2. *We say that the ε-*cutting modulus *of a* finite *metric space (X, d_X), denoted by $\Xi(X, \varepsilon)$, is the infimum over $\Xi > 0$ such that the following holds. For every $r > 0$ and every $\mu \in \Gamma(X, r)$, such that $\rho_\mu \in \Delta_{\Xi \cdot r, 1/2}(X)$, there exists $S \subset X$ such that $\Phi_\mu(S) \leq \varepsilon$.*

For an infinite *metric space (X, d_X), we define the ε-cutting modulus $\Xi(X, \varepsilon)$ to be the supremum over $\Xi(X', \varepsilon)$, where $X' \subset X$ ranges over all the finite subsets of X.*