Bourgain's theorem is tight ①

Expander graphs

$G=(V,E)$ $\quad \varphi(G)=\Omega_d(1)$ $\quad \lambda_2(L_G)=\Omega_d(1)$

G - d-regular $\quad d$ - const.

$d\geq 3$

<u>Cl</u> $\forall f: V\to \mathbb{R}$

$$\frac{1}{n^2}\sum_{u,v}(f(u)-f(v))^2 \leq \frac{1}{\lambda_2(G)}\,\frac{1}{nd}\sum_{(u,v)\in E}(f(u)-f(v))^2$$

$\forall f: V\to \mathbb{R}^d$

$$\underbrace{\frac{1}{n^2}\sum_{u,v}\|f(u)-f(v)\|_2^2}_{(*)} \leq \frac{1}{\lambda_2(G)}\,\underbrace{\frac{1}{nd}\sum_{(u,v)\in E}\|f(u)-f(v)\|_2^2}_{(**)}$$

Any embedding of expander into ℓ_2 with short edges is very "compressed".

Suppose that $f: V\to(\mathbb{R}^d,\ell_2)$ has distortion D.

$(*)\geq \Omega(\log_d n)$

$(**)\leq \dfrac{D}{\lambda_2(G)}$ $\quad\Rightarrow D\geq \Omega(\lambda_2(G)\log_d n)=$

$=\Omega_d(\log n)$.

For ℓ_1 the proof is similar.

This negative result can be used positively for partitioning metric spaces using spectral partitioning.

Padded decompositions and applications

Def Let X be a (finite) metric space. A random partition $\mathcal{P}$ is (α, β)-padded for a scale $\Delta > 0$ if

1) all parts have diameter $\leq \Delta$ always
2) $\forall x \in X \quad \Pr[B(x, \alpha\Delta) \subseteq \mathcal{P}(x)] \geq \beta$.

Stronger than LSH, but harder to construct

Th (will prove later)

For every finite metric X, $\forall \Delta > 0$.
$\exists$ a random partition $\mathcal{P}$ which is (α, β)-padded for a scale Δ for $\alpha \leq \alpha_0$ and $\beta \geq \inf_x \left(\frac{|B(x, \Delta/8)|}{|B(x, \Delta)|} \right)^{8\alpha}$

$\alpha = \varepsilon \qquad \beta = n^{-\theta(\varepsilon)} \qquad \alpha = \Theta\left(\frac{1}{\log n}\right) \quad \beta = \Theta(1).$

Multicut

$G=(V,E)$ k pairs $(s_i,t_i)\in V^2$

Goal: remove min number of vertices to disconnect s_i & t_i $\forall i$.

$$OPT = \min_{\substack{d - 0/1\text{-semi-metric on } V: \\ d(s_i,t_i)=1\ \forall i}} \sum_{(u,v)\in E} d(u,v) \geq$$

$$\geq \min_{\substack{d\text{-semi-metric on } V: \\ d(s_i,t_i)=1\ \forall i}} \sum_{(u,v)\in E} d(u,v) =: OPT'$$

<u>Cl</u> OPT' and respective d^* can be computed in poly-time.

<u>Cl</u> $OPT \leq O(\log n) \cdot OPT'$

<u>Cl</u> Can find a cut of the cost $\leq O(\log n)\, OPT'$ in poly-time.

Sample a padded decomposition for $\Delta = 0.1$.

All pairs are cut (small diameter).

$E[\text{cut value}] =$

$$= \sum_{(u,v)\in E} \Pr[(u,v) \text{ is cut}] \leq$$

$$\leq \sum_{(u,v)\in E} \Pr\{B(u, d(u,v)) \not\subseteq \mathcal{P}(u)\} \leq$$

$$\leq \sum_{(u,v)\in E} O(\log n)\, d(u,v) = O(\log n)\, OPT'$$

$$d(u,v) = \delta \ll \frac{1}{\log n}.$$

$$1 - n^{-O(\delta)} \approx O(\delta \log n).$$

Construction of padded decompositions.

Ball carving!

$R \in [\frac{\Delta}{4}; \frac{\Delta}{2}]$ - uniform

carve balls of radius R until X is covered.

Analysis

Diameter condition is obvious.

$\forall t \le \frac{\Delta}{8}$

$$\Pr[B(x,t) \subset \mathcal{P}(x)] \ge E_R \frac{|B(x, R-t)|}{|B(x, R+t)|} \quad (*)$$

$$h(s) = \log |B(x,s)|$$

$$(*) = E_R\, e^{h(R-t)-h(R+t)} \ge$$

$$\ge e^{E_R[h(R-t)-h(R+t)]} =$$

$$= e^{\frac{4}{\Delta}\int_{\Delta/4}^{\Delta/2}(h(R-t)-h(R+t))dR} =$$

$$= e^{\frac{4}{\Delta}\left(\int_{\Delta/4-t}^{\Delta/4+t} h(R)\,dR - \int_{\Delta/2-t}^{\Delta/2+t} h(R)\,dR\right)} \ge$$

$$\ge e^{\frac{8t}{\Delta}\left(h\left(\frac{\Delta}{4}-t\right)-h\left(\frac{\Delta}{2}+t\right)\right)} =$$

$$= \left(\frac{|B(x,\frac{\Delta}{4}-t)|}{|B(x,\frac{\Delta}{2}+t)|}\right)^{\frac{8t}{\Delta}} \ge \left(\frac{|B(x,\Delta/8)|}{|B(x,\Delta)|}\right)^{\frac{8t}{\Delta}}$$

Further applications (next time)

1) $\forall$ n-point X $\exists$ $n^{1-\varepsilon}$-sized subset $X' \subset X$ s.t. X' embeds into ℓ_1 or ℓ_2 with distortion $O(1/\varepsilon)$.
2) Distance oracles.