1

$$\mathrm{OPT} = \min_{\substack{d\text{-cut}\\ \text{semi-metric}\\ \text{on } V}} \frac{\sum_{(u,v)\in E} d(u,v)}{\sum_{(u,v)\in V^2} d(u,v)} \geq \min_{\substack{d\text{-semi-}\\ \text{metric}\\ \text{on } V}} \frac{\sum_{(u,v)\in E} d(u,v)}{\sum_{(u,v)\in V^2} d(u,v)} =: \mathrm{OPT}'$$

Cl OPT' and respective d^* can be found in polynomial time.

Th $\mathrm{OPT} \leq O(\log n) \cdot \mathrm{OPT}'$

Th One can find a cut which has cost $\leq O(\log n) \cdot \mathrm{OPT}'$ in polynomial time.

Plan: 1) d^* is an ℓ_1-metric.

$$d^*(u,v) = \|f(u) - f(v)\|_1 \quad \text{for } f: V \to \mathbb{R}^d.$$

Can get a cut with the cost

$$\leq \frac{\sum_{(u,v)\in E} d^*(u,v)}{\sum_{(u,v)\in V^2} d^*(u,v)} \quad \text{(no loss!)}.$$

2) General d^*: embed into ℓ_1 with distortion $O(\log n)$ and use the above step.

② a) ℓ_1 metric

$$f: V \to \mathbb{R}^d$$

Want: a cut with the cost

$$\frac{\sum_{(u,v)\in E} \|f(u)-f(v)\|_1}{\sum_{(u,v)\in V^2} \|f(u)-f(v)\|_1} \quad (*)$$

Cl $\exists S_1, \dots, S_m \subset V$ $m = poly(n)$ s.t.

$\alpha_1, \dots, \alpha_m > 0$

$$\|f(u)-f(v)\|_1 = \sum_{\substack{i: \\ (u,v) \text{ crosses } S_i}} \alpha_i \,.$$

$$\Rightarrow \exists S_i \quad \text{s.t.} \quad \frac{E(S_i, V \setminus S_i)}{|S_i| \cdot |V \setminus S_i|} \le (*)$$

b) $f: V \to \mathbb{R}^d$ s.t. for $D \geq 1$

$$d(u,v) \le \|f(u)-f(v)\|_1 \le D \cdot d^*(u,v)$$

$$\frac{\sum_{(u,v)\in E} \|f(u)-f(v)\|_1}{\sum_{(u,v)\in V^2} \|f(u)-f(v)\|_1} \le \frac{D \cdot \sum d^*(u,v)}{\sum d^*(u,v)} \le D \cdot OPT!$$

Want: embed a general metric into ℓ_1 with distortion $D = O(\log n)$.

Randomized construction:

$$S_{ij} \subseteq V \qquad 1 \le i \le O(\log n)$$
$$1 \le j \le O(\log n)$$

$\Pr[v \in S_{ij}] = 2^{-i}$ - independent between v, i, j.

$$f(v)_{ij} = d^*(v, S_{ij})$$

Upper bound:

$$\|f(u) - f(v)\|_1 = \sum_{ij} |f(u)_{ij} - f(v)_{ij}| \le$$

$$\le d^*(u,v) \cdot O(\log^2 n).$$

deterministically

Enough to show:

$$\forall u,v \quad \Pr\left[\|f(u) - f(v)\|_1 \ge \Omega(\log n) \cdot d^*(u,v)\right] \ge$$
$$\ge 1 - \frac{1}{10n^2}.$$

then union bound

$$f = f_1 \oplus f_2 \oplus \dots \oplus f_{O(\log n)}$$

$$f_1(u)_i = d^*(u, S_{i1}) \dots$$

Enough to show:

$$\forall u,v \quad \Pr\left[\|f_1(u) - f_1(v)\|_1 \geq \Omega(1)\cdot d^*(u,v)\right] \geq \Omega(1).$$

then Chernoff.

Now we have one random subset per scale. S_i (formerly S_{i1}).

$|d(u,S) - d(v,S)| \geq r$ iff

$\exists\, t \geq 0$ s.t. $B(u,t) \cap S \neq \emptyset \ \wedge\ B(v,t+r) \cap S = \emptyset$

or $B(u,t+r) \cap S = \emptyset \ \wedge\ B(v,t) \neq \emptyset$.

<u>Lm</u> If $B(u,t) \cap B(v,t+r) = \emptyset$, then

and $|B(u,t)| \geq 2^{i-O(1)}$

$|B(v,t+r)| \leq 2^{i+O(1)}$

$\Pr\left[|d(u,S_i) - d(v,S_i)| \geq r\right] \geq \Omega(1)$

$$s_i = \min r \ \text{ s.t. } \ |B(u,s_i)| \geq 2^i \ \text{ and } \ |B(v,s_i)| \geq 2^i$$

$$\Pr_{S_i}\left[\,|d(u,S_i) - d(v,S_i)| \geq s_{i+1} - s_i\,\right] \geq \Omega(1).$$