(1)

$$\lim_{n\to\infty} \Pr\left[|S_n| \geq t\sqrt{n}\right] = \sqrt{\frac{2}{\pi}} \int_t^{\infty} e^{-x^2/2}\,dx \leq$$

$$\leq \sqrt{\frac{2}{\pi}} \cdot \frac{1}{t} e^{-t^2/2}$$

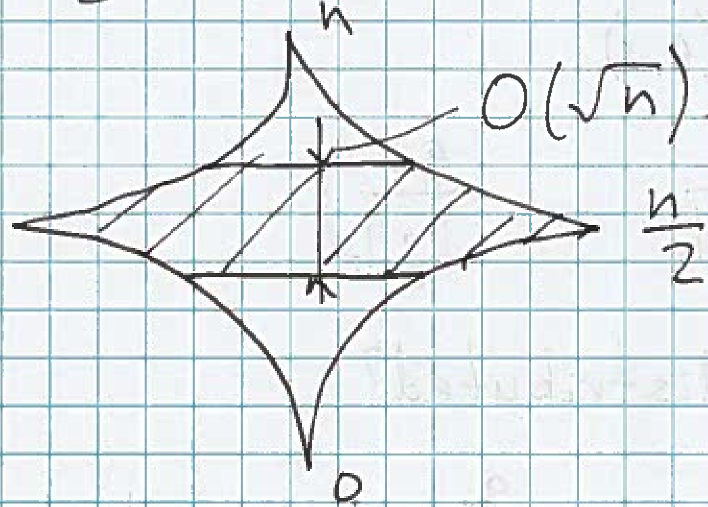
$\geq 3\sqrt{n}$ 0.002

## Geometry

1) $\{0,1\}^n$

Hamming weight of a random point – $\frac{n+S_n}{2}$.

99% points have Hamming weight $\frac{n}{2} \pm O(\sqrt{n})$.



2) $S^{d-1} = \{x \in \mathbb{R}^d \mid \|x\|_2 = 1\}$.

can be parametrized by d−1 numbers.

uniform distribution over $S^{d-1}$ : $\mu(.)$

Can be defined by:

1) $\mu(S^{d-1}) = 1$

2) $\forall$ measurable $A \subset S^{d-1}$
$\forall$ orthogonal $U \in \mathbb{R}^{d \times d}$
$\mu(A) = \mu(UA)$.

How to sample?

$g_1, \dots, g_d \sim \mathcal{N}(0,1)$.

output $\left(\frac{g_1}{\|g\|_2}, \dots, \frac{g_d}{\|g\|_2}\right)$.

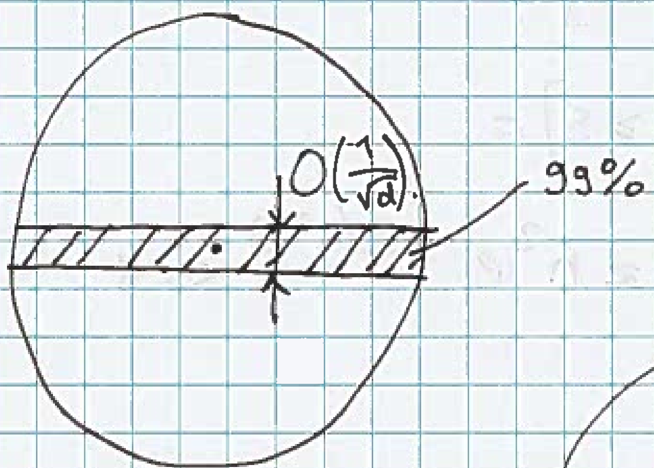
How is $x_1$ distributed?

$$x_1 = \frac{g_1}{\sqrt{g_1^2 + \dots + g_d^2}} = \frac{g_1}{\chi(d)} \quad \text{dependent}$$

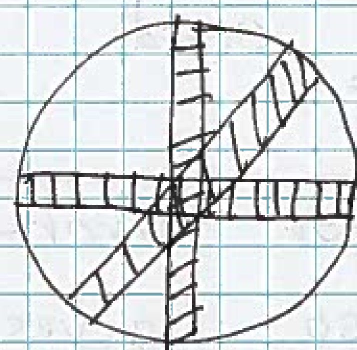
$$\approx \frac{\mathcal{N}(0,1)}{\sqrt{d}}, \text{ since } \chi(d) \text{ is}$$

sharply concentrated around $\sqrt{d}$.

$$\Pr\left[|x_1| \geq \frac{3}{\sqrt{d}}\right] \text{ is very small.}$$




true for any equator.



counter-intuitive!

Application
$\mathbf{x}_1, \dots, \mathbf{x}_k \in S^{d-1}$
$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0.$

Cl $k \leq d.$
What if $|\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \leq \varepsilon$? ($\varepsilon > 0$ is a small parameter).

Lm $k$ can be $2^{\Omega(\varepsilon^2 d)}$!!!
Exponentially many near-orthogonal vectors!

<u>Pf</u> probabilistic method

$n$ random vectors

$$\Pr\left[\exists\, i\, j : |\langle u_i, u_j \rangle| \geq \varepsilon\right] \leq$$

$$\leq n^2 \cdot \Pr\left[|\langle u_1, u_2 \rangle| \geq \varepsilon\right] =$$

$$= n^2 \cdot \Pr\left[|x_1| \geq \varepsilon\right] \approx n^2 e^{-\Omega(\varepsilon^2 d)} \ll 1$$

if $n = 2^{O(\varepsilon^2 d)}$.

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Useful for error-correcting codes, but also gives JL for a regular simplex.

<u>Lm</u> (JL for simplex).

$n$ equidistant points in $\mathbb{R}^{n-1}$.

Can reduce dimension to $O\left(\frac{\log n}{\varepsilon^2}\right)$.