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Lecture 17 – Metric Embeddings II: Sparsest Cut

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1 Main Problem: Sparsest Cut

Problem Given a graph: $G = (V, E)$.Our goal is to minimize the **sparsity** of the graph:

$$\Phi(G) := \frac{E(S, \bar{S})}{|S| \cdot |\bar{S}|}$$

where $S \subset V$ is a non-empty proper subset of vertex set V. And $E(S, \bar{S})$ denotes the sum of the weights of the edges across the cut as:

$$E(S, \bar{S}) = \sum_{e, \textbf{edge e crosses the cut}} w(e)$$

Without mentioned,we will set the weights of the edges as $w(e) = 1$ everywhere.

We'll prove the following theorem:

Theorem 1. *There exists a polynomial time algorithm outputting $S \subset V$ such that:*

$$\frac{E(S, \bar{S})}{|S| \cdot |\bar{S}|} \leq O(\log n) \min_{S^* \subset V} \frac{E(S^*, \bar{S^*})}{|S^*| \cdot |\bar{S^*}|}$$

We denote $OPT := \min_{S^* \subset V} \frac{E(S^*, \bar{S^*})}{|S^*| \cdot |\bar{S^*}|}$ as the minimum sparsity of the graph.

Here's the blueprint for proving this theorem:

$$\text{Graph} \xrightarrow[\textit{Programming}]{\textit{Linear}} \text{Metric Space} \xrightarrow{\text{Bourgain's Thm}} l_1 \xrightarrow{\text{simple}} \textit{cut}$$

2 Preparation1: Semimetrics

Firstly,we will define the concept of "semimetric" (or pseudometric):

Definition 2. *A **semimetric** on X is a function: $d : X \times X \rightarrow R$ such that:*

$$1)d(x,y) \geq 0;$$
$$2)d(x,x) = 0; (\textit{Replace the identity of indiscernibles for metrics})$$
$$3)d(x,y) = d(y,x);$$
$$4)d(x,z) \leq d(x,y) + d(y,z);$$

Now,for a cut presented by vertex set S,we'll have the following function is a semimetric (by easily checking each condition):

Definition 3. *For a graph $G = (V, E)$ and a cut presented by a vertex set S,function $d_S : V \times V \to R$ is defined as follows.*

$$d_S(u,v) = \begin{cases} 1 & \textit{u,v are separated by cut } S \\ 0 & \textbf{\textit{otherwise}} \end{cases}$$

And the function d_S is called ***the semimetric of the cut*** *S.*

Now based on the concept of semimetrics on the graph,we define a more general "optimal"

Definition 4. *We denote OPT^* as the minimum of the following function (d is chosen from the semimetrics on the graph):*

$$OPT^* := \min_{d-semimetric\ on\ V} \frac{\sum_{(u,v)=e\in E} d(u,v)}{\sum_{u,v} d(u,v)}$$

And:

$$d^* := \arg\min_{d-semimetric\ on\ V} \frac{\sum_{(u,v)=e\in E} d(u,v)}{\sum_{u,v} d(u,v)}$$

Here,d^* may not be unique.

Then,we'll have two claims:

Claim 5. *$OPT \geq OPT^*$*

This claim can be deduced from the statement that for every cut $V = S \cup \bar{S}$,d_S is a semimetric on the graph G.

Claim 6. *Given a graph $G = (V, E)$,we can compute OPT^* and a d^* in polynomial time.*

Proof. We can prove this claim by showing the problem can be turned into a linear programming problem:

That is,for $n(n-1)/2, (n = |V|)$ variables $d(u,v), (u \in V, v \in V, u \neq v)$.We are going to calculate :

$$\min \sum_{(u,v)=e\in E} d(u,v)$$

under the constraints:

$$\begin{aligned} &1) d(u,v) \geq 0 \\ &2) d(u,v) + d(v,w) \geq d(u,w) \\ &3) \sum_{u,v} d(u,v) = 1 \end{aligned}$$

And using the result from linear programming we'll get we can solve this problem in polynomial time. □

3 Preparation 2: What can we get after embedding into l_1?

We claimed in our blueprint that after we embed the (semi)metric on the graph into l_1 we can simply get the cut we want. We'll show this fact in this section.

From section 2,we have shown that we can get d^* and OPT^* in polynomial time. And start by this result.We'll prove the following claims:

Claim 7 (Claims for embedding without distortion)**.** *Assume that* $d^*(u,v) = ||f(u) - f(v)||_1$ *for some* $f : V \to R^d$*.Then there exists some* $S \subset V$*,such that:*

$$\frac{E(S,\bar{S})}{|S| \cdot |\bar{S}|} \leq OPT^* \leq OPT$$

Also,we can find an polynomial time algorithm outputting such S*.*

Proof. We'll use the following two results:

1) There exists a set of weights $\omega_S \geq 0$,such that:

$$||f(u) - f(v)||_1 = \sum_{S \subset V} \omega_S d_S(u,v)$$

holds for every $u, v \in V$

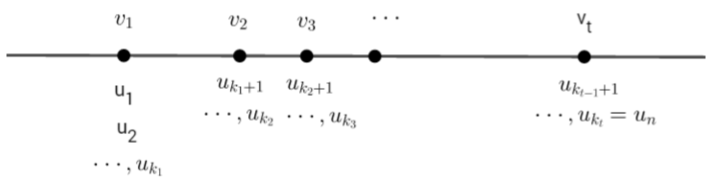
We can prove this statement in two steps:

1.1)It suffices to show that the statement holds for $d = 1$. It is because we can easily decompose R^d space into the straight sum of d R^1 space and then summing up the result as we are using the l_1 norm for $f(u)$.

1.2)Now for the graph G.We sort the vertex of the graph in the way that their corresponding $f(u)$ are listed in a ascending order.That is:

$$\begin{aligned} &f(u_1) = f(u_2) = \cdots f(u_{k_1}) = v_1 \\ &< f(u_{k_1+1}) = \cdots f(u_{k_2}) = v_2 \\ &< \cdots \\ &< f(u_{k_p}) = \cdots f(u_n) = v_t, n = |V|, k_t = n \end{aligned}$$

We can show the list using the figure below.



Now,we claim that for any $u, v \in V$,we'll have:

$$||f(u) - f(v)||_1 = \sum_{r=1}^{t-1} |v_{r+1} - v_r| \cdot d_{\{u_1,...u_{k_r}\}}(u, v)$$

Follows from for any $p, q \in \{1, ..., t\}$,we assume $p < q$ then we can decompose $||v_q - v_p|| = \sum_{r=p}^{q-1} |v_{r+1} - v_r|$,and noticing that $d_{\{u_1,...,u_{k_r}\}}(u, v) = 1$,for $f(u) \leq f(u_{k_r}) \leq f(v), f(u) \neq f(v)$.

2) For any series of number: $\omega_i \geq 0, A_i, B_i \geq 0, i \in \{1, ..., n\}$.We have such an $k \in \{1, ..., n\}$ exists,that k satisfies:

$$\frac{A_k}{B_k} \leq \frac{\sum_i \omega_i A_i}{\sum_i \omega_i B_i}$$

Now from the first statement,we can represent OPT^* as following:

$$\begin{aligned} OPT^* &= \frac{\sum_{(u,v)\in E} ||f(u) - f(v)||}{\sum_{u,v} ||f(u) - f(v)||} \\ &= \frac{\sum_S \omega_S \sum_{(u,v)\in E} d_S(u, v)}{\sum_S \omega_S \sum_{u,v} d_S(u, v)} \end{aligned}$$

And applying the second statement to this equation we will have:

$$\exists S, \frac{\sum_{(u,v)\in E} d_S(u, v)}{\sum_{(u,v)} d_S(u, v)} \leq OPT^*$$

Thus we've accomplished the proof.

□

Remark:We can easily deduce an polynomial time algorithm outputting the S from the prove.Simply speaking,the algorithm takes three steps:

i)Calculating all the coordinate of function f;

ii)Sorting the value of every vertex by the coordinate of function f;

iii)Try all "prefixes cuttings".

If we embed the semimetric on the graph into l_1 with distortion,we can fix the claim above into:

Claim 8 (Claims for embedding with distortion)**.** *Assume that $d^*(u,v) \leq ||f(u) - f(v)||_1 \leq D \cdot d^*(u,v)$ for some $f : V \to R^d$ and a constant D (that is embedding with distortion D).Then there exists some $S \subset V$,such that:*

$$\frac{E(S,\bar{S})}{|S| \cdot |\bar{S}|} \leq D \cdot OPT^* \leq D \cdot OPT$$

Also,we can find an polynomial time algorithm outputting such S.

Proof. We can just follow the similar steps for proving **Claim 7**.Just using the following property:

$$\frac{\sum_{(u,v)\in E} ||f(u) - f(v)||_1}{\sum_{u,v} ||f(u) - f(v)||_1} \leq \frac{D \cdot \sum_{(u,v)\in E} d^*(u,v)}{\sum_{u,v} d^*(u,v)} = D \cdot OPT^*$$

□

4 Metric Embedding: Bourgain's Theorem

Hence now the key point is that how to embed the semimetric on the graph into l_1 norm with as small distortion as possible.Actually,we have the following theorem:

Theorem 9 (Bourgain's Theorem)**.** *For every X is a n points (semi)metric space,then there exists a function: $f : X \to R^{O(\log^2 n)}$ s.t. f is an embedding with distortion $O(\log n)$.*

And we can calculate this mapping in polynomial time.

*Remark:*We will see that the distortion $O(\log n)$ is tight.

Now we will introduce Frechet Embeddings

Definition 10 (Frechet Embeddings)**.** *For d **nonempty** subsets of X: $S_1 \subset X$, $S_2 \subset X$,...,$S_d \subset X$. We define a function:$f : X \to R^d$ as* Frechet Embeddings*:*

$$f(x) = (f_{S_1}(x), \cdots, f_{S_d}(x))$$

Where $f_{S_i}(x) := dist(x, S_i) = \min_{\bar{x}\in S} d_X(x, \bar{x})$.

Here,we have two claims:

Claim 11.

$$\forall S, |f_S(u) - f_S(v)| \leq d_X(u,v)$$

This can be proved by using the triangle inequality for semimetrics.And applying this claim for each dimension of $f(x)$,we'll have:

Claim 12.

$$||f(u) - f(v)||_1 \leq d \cdot d_X(u,v)$$

Now we construct a series of sets as following:

For $1 \leq i \leq \lfloor \log_2 n \rfloor, 1 \leq j \leq O(\log n)$.We decide whether $x \in S_{ij}$ by applying an independent "pick-up" with the possibility $Pr[x \in S_{ij}] = 2^{-i}$

Then applying **Claim 12**,we will have:

Lemma 13. *if* $g(x) = (f_{S_{ij}(x)})_{i,j}$*,then:*

$$||g(u) - g(v)||_1 \leq O(\log^2 n) d_X(u, v)$$

As this is the result for the upper bound for mapping g,we also need the estimate for lower bound of g.

We notice that if we want $|f_S(u) - f_S(v)| \geq \delta$,we must think of the r satisfies $f_S(u) \leq r$ and $f_S(v) \geq r + \delta$.Thus we have:

Lemma 14. *If there exists* r *satisfies* $|B(u, r)| = K$ *(*$|B(u, r)| =$ *number of points in* X *with distance from* u *less than* r*),and* $|B(v, r + \delta)| \in [k/100, 100k]$*.We will have:*

$$Pr[|f_{S_{\log k,j}(u)} - f_{S_{\log k,j}(v)}| \geq \delta] \geq \Omega(1)$$

5 Reference

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