COMS W6998-5: Algorithms through Geometric Lens (Fall'17)	Oct. 16, 2017

Lecture 12 – Graph Coloring

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In last lecture, we said that checking whether G is 3-colorable is in NP-complete. So, today we start with graph G which is 3-colorable, and then try to color G in minimum number of colors in polynomial-time.

Theorem 1 (Kigderson'83)**.** *Suppose a 3-colorable graph G has n vertexes, one can color G in $O(n^{1/2})$ colors.*

Proof. The algorithm process as below:

1. Find node v of highest degree. Suppose the highest degree is Δ.
2. If $\Delta \leq n^{1/2}$, then we can color all graph into $\Delta + 1 = O(n^{1/2})$ color by greedy.
3. If $\Delta > n^{1/2}$, the neighbors of v must be bipartite because of the 3-colorable. Call the two groups in bipartite sub-graph as A, B. Then color v, A, B using 3 new colors.
4. Remove v and its neighbors. Repeat on the rest of the graph.

The total number of colors used: $O(n^{1/2}) + 3\#iterations = O(n^{1/2})$. □

Theorem 2 (KMS theorem)**.** *Suppose a 3-colorable graph G has n vertexes, one can color G in $O(n^{1/3} \log n)$ colors.*

Corollary 3. *Can color G in $O(n^{1/4} \log n)$ colors.*

Proof.

1. While maximum degree $\Delta > n^{3/4}$, just perform Wigderson steps. This uses $3\frac{n}{n^{3/4}} = O(n^{1/4})$ colors.
2. Once maximum degree $\Delta \leq n^{3/4}$, apply KMS Theorem. This gives additional $O(\Delta^{1/3} \log n) = O(n^{1/4} \log n)$ colors.

□

Theorem 4 (Kawabayashi-Thorup' 17)**.** *Can color G in $O(n^{0.19996})$ colors.*

Relaxing:

1. Assign an unknown X_i to each node.(encode color).
2. Impose condition that $\forall(i, j)$ edge, X_i and X_j should be encode with different colors.

Think of $X_i \in R^n$ vector:(in MAX-CUT, $X_i X_j = -1$)

An attempt: "different" $\equiv [X_i \cdot X_j = 0]$
in R^n, can take all X_i to be perpendicular to each other.
in R^3, can take only 3 mutually perpendicular vectors.
The the 3-coloring problem becomes:

$$\begin{cases} X_i \in R^3 \\ \forall (i,j) edge : X_i \cdot X_j = 0 \end{cases}$$

But SDP does not allow "$X_i \in R^3$"

KMS suggestion: "different color" $\triangleq$ X_i and X_j are with angle $120°$. Then we get SDP for 3-coloring:

$$\begin{cases} M_{ij} =< X_i, X_j >, & \forall i,j = 1,...,n \\ M_{ij} = -1/2(\text{det product of 2 vectors at } \deg 120), & \forall (i,j) edge \\ M_{ii} = 1(\|X_i\| = 1) \end{cases}$$

Claim 5. *If $M_{ii} = 1, \forall i \neq j, M_{ij} = -\frac{1}{2}$, there is M is not PSD.*

Definition 6. *Set I of nodes is independent set, if no edge inside I.*

Lemma 7. *can find an independent set of size $\Omega(\frac{n}{\Delta^{1/3}\log n})$.(From SDP solution.)*

The lemma will be proved after we prove the KMS theorem.

Now we prove the KMS theorem:

proof of Theorem 2. Algorithm:

1. Use Lemma 7 to find set I. Color it by a new color.
2. Remove I from graph.
3. If the remain size $< \Delta^{1/3}$, stop and color everything using $< \Delta^{1/3}$ new colors.
4. Otherwise, repeat.

The number of colors used is $\Delta^{1/3} + \#iterations$

In 1 iteration, n vertexes will reduce to $\leq n - C\frac{n}{\Delta^{1/3}\log N} \leq n \cdot (1 - \frac{C}{\Delta^{1/3}\log N})$.

After t iterations, the size of graph will be $\leq N(1 - \frac{C}{\Delta^{1/3}\log N})^t \leq N \cdot e^{-\frac{Ct}{\Delta^{1/3}\log N}}$

For $t = \frac{1}{C}\Delta^{1/3}\log N \log N$, the size is < 1. □

Now we prove the lemma:

proof of Lemma 7. bunch of $X_i \in R^n, \|X_i\| = 1$, $\forall (i,j)$ edge, there is $X_i \cdot X_j = -1/2$.

Partitioning:
- Pick g randomly.
- Let $V(t) = \{i :< X_i, g >\geq t\}$ (t: fix later.)

The idea is same as "ball carving LSH"(family 2)

Then $I = \{i \in V : i$ has no neighbors in $V\}$

Analysis: $\mathcal{E}[I] \geq \Omega(\frac{n}{\Delta^{1/3} \log n})$

1) $\mathcal{E}[V(t)] = \mathcal{E}[\sum_i \chi[i \in V]] = n \cdot Pr[< v, g > \geq t]$

2) upper bound $Pr[< g, u > \geq t | < g, v > \geq t]$ for u and v s.t. $< u, v >= -1/2$ $\square$